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# On the Continuous Cohomology of the Lie Algebra of Vector Fields Associated with Non-Trivial Coefficients (Global Analysis)

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CITATION:

TSUJISHITA, TORU. On the Continuous Cohomology of the Lie Algebra of Vector Fields Associated with Non-Trivial Coefficients (Global Analysis). 数理解析研究所講究録 1977, 291: 1-12

ISSUE DATE:

1977-03

URL:

<http://hdl.handle.net/2433/106172>

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ON THE CONTINUOUS COHOMOLOGY OF THE LIE ALGEBRA OF  
VECTOR FIELDS ASSOCIATED WITH NON-TRIVIAL COEFFICIENTS

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§1. Let  $M$  be a smooth manifold and  $L_M$  the topological Lie algebra of all smooth vector fields on  $M$ . Recently Haefliger ([4]) proved the Bott conjecture, which states that the continuous cohomology of  $L_M$  with trivial coefficients is isomorphic to the singular cohomology of the space of cross-sections of a certain fibre bundle over  $M$ . As for the case associated with the Lie derivative action on a tensor space  $A$  on  $M$ , Losik ([5], [7]) has computed the cohomology of a certain subcomplex (called diagonal) of the standard cochain complex, and Reshetnikov ([9]) has announced partial results concerning the total continuous cohomology  $H^*(L_M, A)$ . In this note we state a theorem which reduces essentially the calculation of  $H^*(L_M, A)$  to that of the diagonal cohomology  $H_{\Delta}^*(L_M, A)$  and the Gelfand-Fuks cohomology  $H^*(L_M)$ .

Details will be published elsewhere.

§2. Let  $W$  be a topological  $L_M$ -algebra.

Let  $C^p(L_M, W)$  ( $p > 0$ ) be the space of all continuous alternating  $p$ -forms on  $L_M$  with values in  $W$  and  $C^0(L_M, W) = W$ . For  $\omega \in C^p(L_M, W)$  ( $p \geq 1$ ), we define  $d\omega \in C^{p+1}(L_M, W)$  by

$$\begin{aligned}
d\omega(x_1, \dots, x_{p+1}) &= \sum_{i=1}^{p+1} (-1)^i x_i \omega(x_1, \dots, \hat{x}_i, \dots, x_{p+1}) \\
&+ \sum_{i < j} (-1)^{i+j-1} \omega([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \\
&\qquad\qquad\qquad \hat{x}_j, \dots, x_{p+1})
\end{aligned}$$

$(x_1, \dots, x_{p+1} \in L_M)$ , and for  $\omega \in C^0(L_M, W) = W$ ,  $d\omega(x) = x\omega$  ( $x \in L_M$ ).

We also define  $\omega \wedge \eta \in C^{p+q}(L_M, W)$  for  $\omega \in C^p(L_M, W)$  and

$\eta \in C^q(L_M, W)$  by

$$\begin{aligned}
&(\omega \wedge \eta)(x_1, \dots, x_{p+q}) \\
&= \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} (-1)^{i_1 + \dots + i_p - \frac{p(p+1)}{2}} \omega(x_{i_1}, \dots, x_{i_p}) \eta(x_{j_1}, \dots, x_{j_q})
\end{aligned}$$

$(x_1, \dots, x_{p+q} \in L_M)$ . Then  $C^*(L_M, W) = \{\oplus C^p(L_M, W), d\}$  turns out to be a commutative differential graded algebra (DG-algebra for short).

Putting  $W = R$ ,  $W = C^\infty(M)$ , we get two DG-algebras  $C^*(L_M, R)$  and  $C^*(L_M, C^\infty(M))$ .

Furthermore, put

$$C_{\Delta}^0(L_M, C^\infty(M)) = C^\infty(M),$$

$$\begin{aligned}
C_{\Delta}^p(L_M, C^\infty(M)) &= \{\omega \in C^p(L_M, C^\infty(M)); \text{supp } \omega(x_1, \dots, x_p) \\
&\subset \bigcap_{i=1}^p \text{supp } x_i (x_1, \dots, x_p \in L_M)\} \quad (p > 0).
\end{aligned}$$

Then  $C_{\Delta}^*(L_M, C^\infty(M)) = \oplus C_{\Delta}^p(L_M, C^\infty(M))$  is a DG-subalgebra of  $C^*(L_M, C^\infty(M))$ .

We note that the de Rham complex  $\Omega_M^*$  of  $M$  can be naturally identified with a DG-subalgebra of  $C_\Delta^*(L, C^\infty(M))$ .

§3. Let  $C^*(L_M, \Omega_M^*) = C^*(L_M, \mathbb{R}) \hat{\otimes} \Omega_M^*$  be the completed tensor product of DG-algebras, which is again a DG-algebra. Just as before we get a DG-subalgebra  $C_\Delta^*(L_M, \Omega_M^*)$  of  $C^*(L_M, \Omega_M^*)$  which consists of support preserving cochains. The inclusion map  $i : \mathbb{R} \hookrightarrow C^\infty(M)$  being an  $L_M$ -homomorphism, there is a DG-algebra homomorphism  $i_* : C^*(L_M, \mathbb{R}) \longrightarrow C^*(L_M, C^\infty(M))$ . Consider  $\kappa = i_* \hat{\otimes} j : C^*(L_M, \Omega_M^*) = C^*(L_M, \mathbb{R}) \hat{\otimes} \Omega_M^* \longrightarrow C^*(L_M, C^\infty(M))$ , where  $j : \Omega_M^* \hookrightarrow C^*(L_M, C^\infty(M))$ . It is easy to see that the image of  $\kappa$  is contained in  $C_\Delta^*(L_M, C^\infty(M))$ . Thus we get the following commutative diagram of DG-algebra homomorphisms :

$$(1) \quad \begin{array}{ccc} C^*(L_M, C^\infty(M)) & \longleftarrow & C^*(L_M, \Omega_M^*) \\ \uparrow & & \uparrow \\ C_\Delta^*(L_M, C^\infty(M)) & \longleftarrow & C_\Delta^*(L_M, \Omega_M^*) \end{array}$$

From this there arises a natural homomorphism of graded algebras :

$$\alpha : \text{Tor}_\Delta^{C^*(L_M, \Omega_M^*)}(C^*(L_M, \Omega_M^*), C_\Delta^*(L_M, C^\infty(M))) \longrightarrow H^*(L_M, C^\infty(M)),$$

where  $\text{Tor}$  denotes the differential torsion functor (cf [1]) and  $H^*(L_M, C^\infty(M))$  the cohomology algebra of the DG-algebra  $C^*(L_M, C^\infty(M))$ .

Theorem I.  $\alpha$  is an isomorphism if  $\dim_{\mathbb{R}} H^*(M, \mathbb{R}) < \infty$ .

§4. We recall the results of Losik ([5]), Guillemin ([3]) and Losik ([6]) and Haefliger ([4]), rewriting in more suitable form for our purpose.

Let  $a(n)$  be the topological Lie algebra of formal vector fields of  $n$ -variables and  $a_0(n)$  the subalgebra of  $a(n)$  consisting of elements without constant terms in the coefficients. We get two DG-algebras  $C^*(a(n), R)$  and  $C^*(a_0(n), R)$  associated with the trivial module  $R$ . Let  $S^*V$  and  $S^*U$  be minimal models for  $C^*(a(n), R)$  and  $C^*(a_0(n), R)$  respectively. Here  $U = R\theta_1 \oplus \dots \oplus R\theta_n$  ( $\deg \theta_i = 2i - 1$ ) and  $S^*U$  is the exterior algebra over  $U$  with trivial differentials. (As for  $S^*V$ , see [4]). Then

Theorem L ([5]). There is a quasi-isomorphism

$$\Omega_M^* \otimes_{\tau} S^*U \longrightarrow C_{\Delta}^*(L_M, C^{\infty}(M))$$

which is  $\Omega_M^*$ -linear. Here  $\Omega_M^* \otimes_{\tau} S^*U$  is the twisted tensor product of  $\Omega_M^*$  and  $S^*U$  defined by the twisting  $\tau(\theta_i) = p_i$ ,  $p_i$  being the  $i$ -th pontrjagin form of  $M$  with respect to a Riemannian metric.

(For the notion of twisted tensor product, see [4].)

Recall that a DG-algebra homomorphism is said to be a quasi-isomorphism if it induces an isomorphism on cohomology level.

Theorem GL ([3], [6]). There are a twisted tensor product

$\Omega_M^* \otimes_{\sigma} S^*V$  and a quasi-isomorphism

$$\beta : \Omega_M^* \otimes_{\sigma} S^*V \longrightarrow C_{\Delta}^*(L_M, \Omega_M^*),$$

which is  $\Omega_M^*$ -linear.

Let  $\Omega_M^* \otimes V$  be the graded vector space such that  $\deg(\omega \otimes v) = -\deg \omega + \deg v$ . Let  $S^*(\Omega_M^* \otimes V)$  be the graded algebra of graded commutative continuous forms on  $\Omega_M^* \otimes V$ .

Theorem H ([4]). There are a DG-algebra structure on

$S^*(\Omega_M^* \otimes V)$  and a quasi-isomorphism

$$\gamma : S^*(\Omega_M^* \otimes V) \longrightarrow C^*(L_M, \mathbb{R}).$$

Let  $\varepsilon : \Omega_M^* \otimes_{\sigma} S^*V \longrightarrow \Omega_M^* \hat{\otimes} S^*(\Omega_M^* \otimes V)$  be the algebraic evaluation map defined in [4]. Let  $\lambda : S^*V \longrightarrow S^*U$  be the DG-algebra homomorphism corresponding to

$$\iota^* : C^*(a(n), \mathbb{R}) \longrightarrow C^*(a_0(n), \mathbb{R})$$

induced by the inclusion  $\iota : a_0(n) \hookrightarrow a(n)$ .

Remark. It is easy to see  $\lambda(S^1V) = 0$ .

Lemma 1. We have the following commutative diagram of DG-

algebra homomorphisms:

$$\begin{array}{ccccc} \Omega_M^* \hat{\otimes} S^*(\Omega_M^* \otimes V) & \xleftarrow{\varepsilon} & \Omega_M^* \otimes_{\sigma} S^*V & \xrightarrow{\text{id} \otimes \lambda} & \Omega_M^* \otimes_{\tau} S^*U \\ \downarrow \text{id} \hat{\otimes} \gamma & & \downarrow \beta & & \downarrow \alpha \\ \Omega_M^* \hat{\otimes} C^*(L_M, \mathbb{R}) & \xleftarrow{\quad} & C_{\Delta}^*(L_M, \Omega_M^*) & \longrightarrow & C_{\Delta}^*(L_M, C^{\infty}(M)). \end{array}$$

Recall the following

Proposition ([1]). Suppose the following commutative diagram of DG-algebra homomorphisms is given:

$$(2) \quad \begin{array}{ccccc} M & \xleftarrow{\quad} & U & \xrightarrow{\quad} & N \\ \lambda \downarrow & & \mu \downarrow & & \downarrow \nu \\ M' & \xleftarrow{\quad} & U' & \xrightarrow{\quad} & N' \end{array}$$

where  $\lambda$ ,  $\mu$  and  $\nu$  are quasi-isomorphisms. Then the induced map  $\text{Tor}^U(M, N) \longrightarrow \text{Tor}^{U'}(M', N')$  is an isomorphism.

Thus we get

Theorem I'. There is an isomorphism of graded algebras:

$$H^*(L_M, C^\infty(M)) \cong \text{Tor}_{M\sigma}^{\Omega_M^* \otimes S^*V} (\Omega_M^* \hat{\otimes} S^*(\Omega_M^* \otimes V), \Omega_M \otimes_{\tau} S^*U).$$

§5. We give a geometric paraphrase of Theorem I'.

Let  $B$  be the principal  $U(n)$ -bundle associated to the complexification of the real tangent bundle of  $M$ . Let  $\hat{E}U_n$  be the restriction of the universal principal  $U(n)$ -bundle to the  $2n$ -skelton of the base  $BU_n$  with respect to the cell decomposition with even-dimensional cells. Put  $E = B \times_{U(n)} \hat{E}U_n$ . Fixing a fibre inclusion mapping  $U(n) \hookrightarrow \hat{E}U_n$ , we get an inclusion mapping:  $B \hookrightarrow E$ . Let  $\Gamma(E)$  be the space of all continuous sections of  $E$  with the compact open topology. Let  $\varepsilon : M \times \Gamma(E) \longrightarrow E$  be the evaluation mapping.

Let  $X \mapsto A^*(X)$  be any contravariant functor which associates to each topological space a commutative DG-algebra  $A^*(X)$  over  $R$  such that  $H(A^*(X)) = H^*(X, R)$  (cf. [11]). Corresponding to the diagram of topological spaces:

$$M \times \Gamma(E) \xrightarrow{\epsilon} E \longleftarrow B$$

We get a diagram of DG-algebras:

$$A^*(M \times \Gamma(E)) \longleftarrow A^*E \longrightarrow A^*B.$$

We say that two triples of DG-algebras  $T = \{M \longleftarrow U \longrightarrow N\}$  and  $T' = \{M' \longleftarrow U' \longrightarrow N'\}$  are equivalent if there is a sequence of triples  $T_0 = T, T_1, \dots, T_{n-1}, T_n = T'$  such that for each  $i$  ( $0 \leq i \leq n-1$ ) there is a quasi-isomorphism  $T_i \longrightarrow T_{i+1}$  or  $T_{i+1} \longrightarrow T_i$ . Here, a quasi-isomorphism  $\{M \longleftarrow U \longrightarrow N\} \rightarrow \{M' \longleftarrow U' \longrightarrow N'\}$  is simply a commutative diagram (2) such that  $\lambda, \mu$  and  $\nu$  are quasi-isomorphisms. Note that if  $T$  and  $T'$  are equivalent then  $\text{Tor}^U(M, N) = \text{Tor}^{U'}(M', N')$ .

Lemma 2. Triples  $T_M = \{ \Omega_M \overset{\Delta}{\otimes} S^*(\Omega_M \otimes V) \longleftarrow \Omega_M \underset{\sigma}{\otimes} S^*V \longrightarrow \Omega_M \underset{\tau}{\otimes} S^*V \}$

and  $\{A^*(M \times \Gamma(E)) \longleftarrow A^*E \longrightarrow A^*B\}$  are equivalent.

On the other hand, we can show the following

Lemma 3.  $\epsilon : M \times \Gamma(E) \longrightarrow E$  is a Serre fibering.

Recall the following

Theorem (Eilenberg-Moore-Gugenheim [1], [2].) Let  $X \longrightarrow E$  be a Serre fibering and  $\iota : B \longrightarrow E$  a mapping. Let  $Y = \iota^*X$  be the induced fibering. Assume that  $\pi_1(E) = 0$ . Then we have an isomorphism of graded algebras:



$$\mathrm{Tor}^{A^*E}(A^*(M \times \Gamma(E)), A^*B) \cong H^*(Y, \mathbb{R}).$$

Let  $Y$  be the fibering over  $B$  induced from  $\varepsilon : M \times \Gamma(E) \longrightarrow E$  by  $B \hookrightarrow E$ . Then, in view of  $\pi_1(\hat{E}U_n) = 0$ , we have the following

Theorem II. If  $\dim_{\mathbb{R}} H^*(M, \mathbb{R}) < \infty$  and  $\pi_1(M) = 0$ , then

$$H^*(L_M, C^\infty(M)) \cong H^*(Y, \mathbb{R}).$$

§6. We consider examples.

Let  $M = \mathbb{R}^n$ . Since  $\mathbb{R} \hookrightarrow \Omega_{\mathbb{R}^n}^*$  is a quasi-isomorphism, it is easy to see that the triple  $T_{\mathbb{R}^n}$  is equivalent to  $\{S^*V \xleftarrow{\mathrm{id}} S^*V \longrightarrow S^*U\}$ . Hence

$$H^*(L_{\mathbb{R}^n}, C^\infty(\mathbb{R}^n)) \cong \mathrm{Tor}^{S^*V}(S^*V, S^*U) \cong S^*U.$$

Thus

$$\text{Corollary 1. } H^*(L_{\mathbb{R}^n}, C^\infty(\mathbb{R}^n)) \cong H_{\Delta}^*(L_{\mathbb{R}^n}, C^\infty(\mathbb{R}^n)) \cong S^*U.$$

Let  $M = S^1$ . Then it is easy to see that the triple  $T_{S^1}$  can be replaced by

$$T' = \{S^*(t, \sigma, \xi) \xleftarrow{\alpha} S^*(t, \sigma) \xrightarrow{\beta} S^*(t, \theta)\}$$

where  $\deg t = \deg \theta = 1$ ,  $\deg \sigma = 3$ ,  $\deg \xi = 2$ ,  $dt = d\theta = d\sigma = d\xi = 0$ ,  $\alpha(t) = t$ ,  $\alpha(\sigma) = t\xi + \sigma$ ,  $\beta(t) = t$ ,  $\beta(\sigma) = 0$ . Here  $S^*(x, y, \dots)$  denotes the free anti-commutative graded algebra generated by

$x, y, \dots$ . We can check immediately that  $T'$  is equivalent to

$$T'' = \{ S^*(t, \sigma, \xi) \xleftarrow{\tilde{\alpha}} S^*(t, \sigma) \xrightarrow{\beta} S^*(t, \theta) \}$$

where  $\tilde{\alpha}(t) = t, \tilde{\alpha}(\sigma) = \sigma$ . Thus

$$\text{Tor}^{S^*(t, \sigma)}(S^*(t, \sigma, \xi), S^*(t, \theta)) \cong S^*(t, \theta, \xi).$$

Corollary 2.  $H^*(L_{s^1}, C^\infty(S^1)) \cong S^*(t, \theta, \xi)$ , where

$\deg t = \deg \theta = 1, \deg \xi = 2$ .

§7. Finally we consider the general case.

Let  $G^k (k \geq 1 \dots)$  be the Lie group of  $k$ -jets at the origin of diffeomorphisms of  $\mathbb{R}^n$  fixing 0. Let  $A$  be a finite dimensional real  $G^k$ -module. For a smooth manifold  $M$  of dimension  $n$ , we denote by  $S^k_M$  the  $G^k$ -principal bundle canonically associated to  $M$ . Put  $\alpha = S^k_M \times A$ . Then  $\alpha$  is a  $\text{Diff}(M)$ -bundle over  $M$ .

Hence  $A_M = \Gamma^\infty(\alpha)$  can be naturally regarded as a topological  $L_M$ -module. We can then define  $C^*(L_M, A_M), C^*_\Delta(L_M, A_M)$ , and  $H^*(L_M, A_M)$  just as in §2. The natural pairing  $C^\infty(M) \otimes A_M \longrightarrow A_M$  gives rise to a differential graded  $C^*_\Delta(L_M, C^\infty(M))$ -module structure on  $C^*_\Delta(L_M, A_M)$ . Using the DG-algebra homomorphism  $\kappa : C^*_\Delta(L_M, \Omega_M^*) \longrightarrow C^*_\Delta(L_M, C^\infty(M))$ , we regard  $C^*_\Delta(L_M, A_M)$  as a differential graded  $C^*_\Delta(L_M, \Omega_M^*)$ -module. On the other hand, the  $G^k$ -module  $A$  gives rise to an  $a_0(n)$ -module  $A$  canonically (cf [10]).

Theorem III. If  $\dim_{\mathbb{R}} H^*(M, \mathbb{R}) < \infty$  and  $\dim_{\mathbb{R}} H^i(a_0(n), A) < \infty$  ( $i = 0, 1, \dots$ ), then there is an isomorphism of graded vector space:

$$H^*(L_M, A_M) \cong \text{Tor}^{C_{\Delta}^*(L_M, \Omega_M^*)}(C^*(L_M, \Omega_M^*), C_{\Delta}^*(L_M, A_M)).$$

Remark. Let  $G^1 \longrightarrow G^k$  be a lifting of  $G^k \longrightarrow G^1$ . Then  $A$  can be regarded as a  $G^1$ -module. If  $A$  is completely reducible  $G^1$ -module, it is easy to see that  $\dim_{\mathbb{R}} H^i(a_0(n), \mathbb{R}) < \infty$  ( $i = 0, 1, 2, \dots$ ).

Under the hypotheses of Theorem II, we have the following corollaries.

Corollary 3. There is a spectral sequence converging to  
 $H^*(L_M, A_M)$  whose  $E_2$ -term is

$$\text{Tor}^{H(C_{\Delta}^*(L_M, \Omega_M^*))}(H(C^*(L_M, \Omega_M^*)), H(C_{\Delta}^*(L_M, A_M))).$$

Corollary 4. If  $H^*(C_{\Delta}^*(L_M, A_M)) = 0$ , then  $H^*(L_M, A_M) = 0$ .  
Especially,  $H^*(L_M, L_M) = 0$ , where  $L_M$  is the  $L_M$ -module defined  
by the adjoint action.

Corollary 5.  $\dim_{\mathbb{R}} H^i(L_M, A_M) < \infty$  ( $i = 0, 1, 2, \dots$ ). (cf [9]).

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